Geometry of 2d spacetime and quantization of particle dynamics

George Jorjadze[†] and Włodzimierz Piechocki[‡]

[†]Razmadze Mathematical Institute, Tbilisi, Georgia

[‡]Sołtan Institute for Nuclear Studies, Warsaw, Poland

Abstract

We analyze classical and quantum dynamics of a particle in 2d spacetimes with constant curvature which are locally isometric but globally different. We show that global symmetries of spacetime specify the symmetries of physical phase-space and the corresponding quantum theory. To quantize the systems we parametrize the physical phase-space by canonical coordinates. Canonical quantization leads to unitary irreducible representations of $SO_\uparrow(2.1)$ group.

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1 Introduction

Global properties of Lorentzian spacetime play an important role in the problem of unification of General Relativity and Quantum Mechanics. We present some results concerning the corresponding problem in the case of 2-dimensional spacetime with constant curvature $R_0 \neq 0$. We consider classical and quantum dynamics of a relativistic particle in the following cases:

(i) spacetime is a one-sheet hyperboloid ($R_0 < 0$),

(ii) spacetime is a stripe ($R_0 > 0$),

(iii) spacetime is a half-plane ($R_0 < 0$ and $R_0 > 0$).

These examples of spacetime are (separately, for $R_0 < 0$ and $R_0 > 0$) locally isometric but have different global properties. We have found the relations among the symmetries of spacetime, physical phase-space and corresponding quantum theory. For completeness of the present paper we recall

some results of [1], where we considered the dynamics of a particle in the Liouville field to test our quantization method for gauge invariant theories. Some physical aspects of particle dynamics in the singular Liouville field were discussed in [2].

Dynamics of a relativistic particle with mass m_0 in gravitational field $g_{\mu\nu}(x^0, x^1)$ is defined by the action

$$S = \int L(\tau)\ d\tau, \qquad L(\tau) := -m_0 \sqrt{g_{\mu\nu}(x^0(\tau), x^1(\tau))\ \dot{x}^\mu(\tau) \dot{x}^\nu(\tau)}, \tag{1.1}$$

where τ is an evolution parameter and $\dot{x}^\mu := dx^\mu/d\tau$. The coordinate x^0 is associated with time and it is assumed that $\dot{x}^0 > 0$.

The action (1.1) is invariant under reparametrization $\tau \rightarrow f(\tau)$. This gauge symmetry leads to the constrained dynamics in the Hamiltonian formulation [3]. The constraint reads

$$\Phi := g^{\mu\nu} p_\mu p_\nu - m_0^2 = 0, \tag{1.2}$$

where $p_\mu := \partial L/\partial \dot{x}^\mu$ are canonical momenta (we use units with $c = 1 = \hbar$).

In what follows we assume that the physical phase-space is the set of particle trajectories in spacetime [4,5]. This set can be identified with the space of gauge invariant dynamical integrals. We parametrize this space by coordinates suitable for canonical quantization.

2 Geometry of 2d manifolds

One can always choose local coordinates on 2d Lorentzian manifold in such a way that the metric tensor $g_{\mu\nu}$ takes the form [6]

$$g_{\mu\nu}(X) = \exp \varphi(X) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.1}$$

where $X := (x^0, x^1)$ and $\varphi(X)$ is a field.

The scalar curvature calculated for (2.1) is given by

$$R(X) = \exp\left(-\varphi(X)\right) \left(\partial_1^2 - \partial_0^2\right)\ \varphi(X). \tag{2.2}$$

Therefore, the case $R(X) = R_0 = const$ leads to the equation for φ

$$(\partial_0^2 - \partial_1^2)\varphi(X) + R_0 \exp\varphi(X) = 0, \tag{2.3}$$

which is known as the Liouville equation [7]. This equation is invariant under the conformal transformations of the metric (2.1)

$$\begin{gathered} x^\pm \longrightarrow y^\pm(x^\pm), \\ \varphi(x^+, x^-) \longrightarrow \tilde{\varphi}(x^+, x^-) := \varphi(y^+(x^+), y^-(x^-)) + \log[{y^+}'(x^+){y^-}'(x^-)], \end{gathered} \tag{2.4}$$

where $x^\pm := x^0 \pm x^1$ and ${y^\pm}' := dy^\pm/dx^\pm$.

The general solution to (2.3) is given by

$$\varphi(x^+, x^-) = \log \frac{8{A_+}'(x^+){A_-}'(x^-)}{|R_0|\,[A_+(x^+) - \epsilon A_-(x^-)]^2}, \tag{2.5}$$

where ${A_\pm}' := dA_\pm/dx^\pm$, $\epsilon := |R_0|/R_0$.

Solution (2.5) is invariant under the transformation (2.4) with

$$y^+(x^+) = A_+^{-1}\left(\frac{aA_+(x^+) + b}{cA_+(x^+) + d}\right), \quad y^-(x^-) = A_-^{-1}\left(\frac{aA_-(x^-) + \epsilon b}{\epsilon c A_-(x^-) + d}\right), \tag{2.6}$$

where A_+^{-1} and A_-^{-1} are the inverse of A_+ and A_- functions, respectively.

Taking $A_\pm(x^\pm) = x^\pm$ we get the solution

$$\varphi(x^+, x^-) = \log \frac{8}{|R_0|(x^+ - \epsilon x^-)^2}. \tag{2.7}$$

The conformal transformation (2.4) leads (2.7) to the general solution (2.5). Thus, we conclude that all Lorentzian 2d manifolds with the same constant curvatures are locally isometric. However, they can be globally different. Eqs.(2.1) and (2.3) do not define the global properties of spacetime manifold. In what follows we consider the models of spacetime with different global properties indicated in the Introduction.

3 Dynamics on hyperboloid

Let (y^0, y^1, y^2) be the standard coordinates on 3d Minkowski space with the metric tensor $\eta_{ab} = diag(+,-,-)$. A one-sheet hyperboloid $\mathbf{H}$ is defined by

$$-(y^0)^2 + (y^1)^2 + (y^2)^2 = m^{-2}, \tag{3.1}$$

where $m > 0$ is a fixed parameter.

Making use of the parametrization

$$y^0 = -\frac{\cot m\rho}{m}, \quad y^1 = \frac{\cos m\theta}{m \sin m\rho}, \quad y^2 = \frac{\sin m\theta}{m \sin m\rho},$$
$$\text{where} \quad \rho \in]0, \pi/m[, \quad \theta \in [0, 2\pi/m[\tag{3.2}$$

we get the induced metric tensor on $\mathbf{H}$

$$g_{\mu\nu}(\rho, \theta) = \frac{1}{\sin^2 m\rho} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.3}$$

which has the conformal form (2.1) with

$$\varphi(\rho, \theta) = -\log \sin^2 m\rho. \tag{3.4}$$

Since $\mathbf{H}$ has a constant curvature $R = -2m^2$ (see [6]), it is clear that (3.4) is the solution of (2.3) for $R_0 = -2m^2$, where the parameters ρ and θ are identified with the spacetime coordinates x^0 and x^1, respectively.

The Lagrangian (1.1) in this case reads

$$L = -m_0 \sqrt{\frac{\dot{\rho}^2 - \dot{\theta}^2}{\sin^2 m\rho}} \ . \tag{3.5}$$

We assume that trajectories are timelike ($|\dot{\rho}| > |\dot{\theta}|$) and $\dot{\rho} > 0$.

The hyperboloid (3.1) is invariant under the Lorentz transformations, i.e., $SO_{\uparrow}(2.1)$ is the symmetry group of our system. The corresponding infinitesimal transformations (rotation and two boosts) are

$$(\rho, \theta) \longrightarrow (\rho, \theta + \alpha_0/m),$$
$$(\rho, \theta) \longrightarrow (\rho - \alpha_1/m \ \sin m\rho \sin m\theta, \theta + \alpha_1/m \ \cos m\rho \cos m\theta),$$
$$(\rho, \theta) \longrightarrow (\rho + \alpha_2/m \ \sin m\rho \cos m\theta, \theta + \alpha_2/m \ \cos m\rho \sin m\theta). \tag{3.6}$$

The dynamical integrals for (3.6) read

$$J_0 = \frac{p_\theta}{m}, \qquad J_1 = -\frac{p_\rho}{m} \sin m\rho \sin m\theta + \frac{p_\theta}{m} \cos m\rho \cos m\theta,$$
$$J_2 = \frac{p_\rho}{m} \sin m\rho \cos m\theta + \frac{p_\theta}{m} \cos m\rho \sin m\theta, \tag{3.7}$$

where $p_\theta := \partial L / \partial \dot{\theta}$, $p_\rho := \partial L / \partial \dot{\rho}$ are canonical momenta.

Since J_0 is connected with space translations (see (3.6)), it defines particle momentum $p_\theta = mJ_0$.

It is clear that the dynamical integrals (3.7) satisfy the commutation relations of $sl(2.\mathbf{R})$ algebra

$$\{J_a, J_b\} = \varepsilon_{abc}\eta^{cd}J_d, \tag{3.8}$$

where η^{cd} is the Minkowski metric tensor and ε_{abc} is the anti-symmetric tensor with $\varepsilon_{012} = 1$.

The mass shell condition (1.2) takes the form

$$\sin^2 m\rho\ (p_\rho^2 - p_\theta^2) = m_0^2, \tag{3.9}$$

which leads to the relation

$$J_0^2 - J_1^2 - J_2^2 = -a^2, \qquad a := \frac{m_0}{m}\ . \tag{3.10}$$

According to (3.1), (3.2) and (3.7) the trajectories satisfy the equations

$$J_a y^a = 0, \qquad y_a y^a = -m^2, \qquad J_1 y_2 - J_2 y_1 = \frac{p_\rho}{m^2}\ . \tag{3.11}$$

Note that $p_\rho < 0$, since $\dot{\rho} > 0$. Then, for a given point (J_0, J_1, J_2) of (3.10), Eq.(3.11) uniquely defines the trajectory on the hyperboloid $\mathbf{H}$. One can check that all points of the hyperboloid (3.10) are available for the dynamics. Therefore, there is one-to-one correspondence between the hyperboloid (3.10) and the space of particles trajectories. Since the physical phase-space is assumed to be the space of trajectories [4,5], we conclude that the hyperboloid (3.10) is the physical phase-space.

To quantize the system we introduce the cylindrical coordinates $J \in \mathbf{R}$ and $\phi \in S^1$, which parametrize the hyperboloid (3.10) by

$$J_0 = J, \qquad J_1 = \sqrt{J^2 + a^2}\cos\phi, \qquad J_2 = \sqrt{J^2 + a^2}\sin\phi. \tag{3.12}$$

The canonical commutation relation $\{J, \phi\} = 1$ leads to (3.8).

In the 'ϕ-representation' the set of functions $\psi_n = \exp in\phi$, $(n \in Z)$ form the basis of the Hilbert space $L_2(S^1)$. For the corresponding quantum operators $\hat{J}_a$ we have to choose the definite operator ordering in (3.12). We specify it by the following requirements:

a) the operators corresponding to (3.12) are self-adjoint,

b) they generate global $SO_{\uparrow}(2.1)$ transformations,

c) the quantum Casimir number is equal to the classical one, i.e., $\hat{C} = -a^2\hat{I}$, where $\hat{C} := \hat{J}_0^2 - \hat{J}_1^2 - \hat{J}_2^2$.

These conditions can be satisfied by the operators

$$\begin{aligned} &\hat{J}_0 = \hat{J} = -i\partial_\phi, \quad \hat{J}_+ = e^{i\phi}\sqrt{\hat{J}^2 + \hat{J} + a^2}, \\ &\hat{J}_- = \sqrt{\hat{J}^2 + \hat{J} + a^2}\; e^{-i\phi}, \end{aligned} \tag{3.13}$$

where $\hat{J}_\pm := \hat{J}_1 \pm i\hat{J}_2$ and we get

$$\begin{aligned} \hat{J}_0\psi_n = n\psi_n, \quad \hat{J}_+\psi_n = \sqrt{n^2+n+a^2}\;\psi_{n+1}, \\ \hat{J}_-\psi_n = \sqrt{n^2-n+a^2}\;\psi_{n-1}, \qquad \hat{C}\psi_n = -a^2\psi_n. \end{aligned} \tag{3.14}$$

The case $a^2 \geq 1/4$ corresponds to the unitary irreducible representation (UIR) of the continuous principal series of $SL(2.\mathbf{R})$ group, whereas the case $0 < a^2 < 1/4$ presents the UIR of the additional continuous series [8].

Due to (3.14), momentum of a quantum particle $\hat{p}_\theta = m\hat{J}_0$ can take only discrete values $P_n = mn$, $(n \in Z)$. This result is related to the fact that the space of the considered spacetime is compact.

4 Dynamics on half-plane ($R_0 < 0$)

Let us consider the solution (2.7) for $\epsilon < 0$. The Lagrangian (1.1) in this case reads

$$L = -2a\sqrt{\frac{\dot{x}^+\dot{x}^-}{(x^+ + x^-)^2}}, \qquad a := \frac{m_0}{m}, \tag{4.1}$$

where $x^\pm := x^0 \pm x^1$ and $m = \sqrt{-R_0/2}$. It is assumed that $\dot{x}^\pm > 0$, which leads to $p_\pm < 0$.

Formally, (4.1) is invariant under the fractional-linear transformations

$$x^+ \to \frac{ax^+ + b}{cx^+ + d}, \qquad x^- \to \frac{ax^- - b}{-cx^- + d}, \qquad ad - bc = 1. \tag{4.2}$$

Thus, formally, $SL(2.\mathbf{R})/\mathbf{Z_2}$ (which is isomorphic to $SO_{\uparrow}(2.1)$) is the symmetry of our system. But, the transformations (4.2) are well defined on the plane only for $c = 0$. The corresponding transformations (with $c = 0$) form the group of dilatations and translations (along x^1), which is a global symmetry of the spacetime.

The infinitesimal transformations for (4.2) are

$$x^{+} \to x^{\pm} \pm \alpha_0, \qquad x^{+} \to x^{\pm} + \alpha_1 x^{\pm}, \qquad x^{+} \to x^{\pm} \pm \alpha_2 (x^{\pm})^2 \tag{4.3}$$

and the corresponding dynamical integrals read

$$P = p_+ - p_-, \qquad K = p_+ x^+ + p_- x^-, \qquad M = p_+ (x^+)^2 - p_- (x^-)^2, \tag{4.4}$$

where $p_\pm = \partial L / \partial \dot{x}^\pm$.

The dynamical integrals (4.4) satisfy again the commutation relations (3.8) with

$$J_0 = \frac{1}{2}(P + M), \qquad J_1 = \frac{1}{2}(P - M), \qquad J_2 = K. \tag{4.5}$$

The constraint equation (1.2) in this case can be rewritten as

$$p_+ p_- (x^+ + x^-)^2 = a^2, \tag{4.6}$$

which leads to

$$K^2 - PM = a^2. \tag{4.7}$$

Eq.(4.4) defines particle trajectories on the plane

$$P x^+ x^- + K(x^+ - x^-) = M \ . \tag{4.8}$$

Eq.(4.1) shows that $x^0 = 0$ is the singularity line in the spacetime. Particle needs infinite proper time to reach it [2]. This is why the dynamics of the particle can be considered for $x^0 < 0$ and $x^0 > 0$ separately. In such interpretation we deal with two independent systems each without singularity [2]. Since the dynamics in both cases is similar, we consider only the case $x^0 > 0$, i.e., spacetime is $\mathbf{R}_+ \times \mathbf{R}$.

Since the half plane and the hyperboloid (3.1) have the same curvature $R = -2m^2$, one can define the isometry map from two half-planes $(\mathbf{R}_- \times \mathbf{R}) \cup (\mathbf{R}_+ \times \mathbf{R})$ to hyperboloid

$$y^0 = \frac{1 - m^2 x^+ x^-}{m^2 (x^+ + x^-)}, \qquad y^1 = \frac{1 + m^2 x^+ x^-}{m^2 (x^+ + x^-)}, \qquad y^2 = \frac{x^+ - x^-}{m(x^+ + x^-)} \ . \tag{4.9}$$

The map (4.9) is invertible and it covers almost all hyperboloid except two generatrices given by $y^0 + y^1 = 0$. Thus, the half-planes can be considered as two different patches of the hyperboloid (3.1).

According to (4.7), $K = \pm a$ for $P = 0$. But since $p_\pm < 0$, the trajectories with $P = 0$ and $K = a$ cannot exist for $x^0 > 0$ (see (4.4)). Any other point (P, K, M) of the hyperboloid uniquely specifies the particle trajectory. Therefore, the physical phase-space is defined by the hyperboloid (4.7) without the line $P = 0, K = a$. Hence, $SO_\uparrow(2.1)$ is not the symmetry group of our classical system. The symmetry transformations of the physical phase-space are translations and dilatations. These transformations are generated by the dynamical integrals P and K, respectively. The transformations generated by M are not defined globally. Thus, the physical phase-space has the same symmetry as the spacetime.

To quantize the system we parametrize the physical phase-space (which is isomorphic to $\mathbf{R^2}$) by the coordinates (p, q) [9]

$$P = p, \qquad K = pq - a, \qquad M = pq^2 - 2aq. \tag{4.10}$$

The canonical commutation relation $\{p, q\} = 1$ provides the commutation relations of $sl(2.\mathbf{R})$ algebra

$$\{P, K\} = P, \qquad \{P, M\} = 2K, \qquad \{K, M\} = M\ . \tag{4.11}$$

Applying the symmetric operator ordering in 'q-representation' we get

$$\hat{P} = -i\partial_q, \qquad \hat{K} = -iq\partial_q - (a + \frac{i}{2}), \qquad \hat{M} = -iq^2\partial_q - 2(a + \frac{i}{2})q, \tag{4.12}$$

which gives the realization of the classical commutation relations (4.11).

According to the quantization principles quantum observables should be represented by self-adjoint operators. The operators $\hat{P}$ and $\hat{K}$ are self-adjoint and they generate the group of translations and dilatations, which is the symmetry group of spacetime. The symmetric ordering in 'q-representation' leads to the symmetric operator $\hat{M}$. The self-adjoint extension of $\hat{M}$ does exist, but it is not unique [1]. This ambiguity can be parametrized by continuous parameter $\alpha \in S^1$. For $\alpha = 0$, we get UIR of $SO_\uparrow(2.1)$ group. The case $\alpha = \pi$ gives UIR of $SL(2.\mathbf{R})$. Other values of α lead to UIR of the universal covering group $\widetilde{SL}(2.\mathbf{R})$. The Casimir number for all these representations is $C = -(a^2 + 1/4)$, but these representations for different α are unitarily non-equivalent.

5 Dynamics on stripe (and half-plane, $R_0 > 0$)

Let us consider the spacetime to be a stripe

$$\mathcal{S} := \{(t,x) \mid t \in \mathbf{R},\ x \in]0, \pi/m[\}, \tag{5.1}$$

with the metric tensor

$$g_{\mu\nu}(t,x) = \frac{1}{\sin^2 mx} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

It defines the spacetime with constant positive curvature $R = 2m^2$.

The Lagrangian (1.1) in this case

$$L = -m_0 \sqrt{\frac{\dot{t}^2 - \dot{x}^2}{\sin^2 mx}} \tag{5.3}$$

is invariant under the action of the universal covering group $\widetilde{SL}(2.\mathbf{R})$. The corresponding infinitesimal transformations are

$$\begin{gathered} (t,x) \longrightarrow (t - \alpha_0/m, x), \\ (t,x) \longrightarrow (t - \alpha_1/m\ \cos mx \cos mt, x + \alpha_1/m\ \sin mx \sin mt), \\ (t,x) \longrightarrow (t - \alpha_2/m\ \cos mx \sin mt, x - \alpha_2/m\ \sin mx \cos mt) \end{gathered} \tag{5.4}$$

and they lead to the dynamical integrals

$$\begin{gathered} J_0 = -\frac{p_t}{m}, \qquad J_1 = -\frac{p_t}{m} \cos mx \cos mt + \frac{p_x}{m} \sin mx \sin mt, \\ J_2 = -\frac{p_t}{m} \cos mx \sin mt - \frac{p_x}{m} \sin mx \cos mt, \end{gathered} \tag{5.5}$$

which satisfy the commutation relations (3.8).

The mass-shell condition (1.2)

$$\sin^2 mx\ (p_t^2 - p_x^2) = m_0^2, \tag{5.6}$$

leads to the relation

$$J_0^2 - J_1^2 - J_2^2 = a^2, \qquad a := \frac{m_0}{m}\ , \tag{5.7}$$

which defines two-sheet hyperboloid.

The physical condition $\dot{t} > 0$ gives $p_t < 0$. Thus, the space of the dynamical variables (5.5) is only the upper-hyperboloid ($J_0 > 0$). According to (5.5) we have

$$J_0 \cos mx = J_1 \cos mt - J_2 \sin mt \tag{5.8}$$

and each point of the upper-hyperboloid defines the trajectory (5.8) uniquely. These trajectories are periodic in time t with the period $2\pi/m$. Hence, the space of trajectories has the symmetry of $SO_{\uparrow}(2.1)$ group, but not of $\widetilde{SL}(2.\mathbf{R})$. Therefore, the upper-hyperboloid describes the space of trajectories and it can be considered as the physical phase-space of the system.

Due to the translation invariance in time particle energy E is conserved and from (5.5) we have $E = mJ_0$. Eq. (5.8) shows that particle oscillates between the 'edges' of space around the stationary point $x = \pi/m$.

For the quantization we use the parametrization

$$J_0 = \frac{1}{2}(p^2+q^2)+a, \quad J_1 = \frac{1}{2}p\,\sqrt{p^2+q^2+4a}, \quad J_2 = \frac{1}{2}q\,\sqrt{p^2+q^2+4a}, \tag{5.9}$$

where (p,q) are coordinates on a plane. It is easy to see that (5.9) defines the unique parametrization of the upper-hyperboloid and the canonical commutation relation $\{p,q\} = 1$ leads to (3.8).

To solve the ordering problem for the operators corresponding to (5.9), we again impose the requirements (see the case of one-sheet hyperboloid):

a) the operators $\hat{J}_a$ corresponding to (5.9) are self-adjoint,

b) they generate $SO_{\uparrow}(2.1)$ global transformations,

c) the quantum Casimir number equals a^2.

Now, these requirements can be satisfied *only* for the *discrete* values of the parameter a

$$a = \sqrt{k(k-1)}, \qquad \text{with} \qquad k = 2,3,4,.... \tag{5.10}$$

The corresponding operators $\hat{J}_a$ are

$$\hat{J}_0 = a^+a^- + k, \qquad \hat{J}_+ = a^+\sqrt{a^+a^- + 2k}, \qquad \hat{J}_- = \sqrt{a^+a^- + 2k}\; a^-, \tag{5.11}$$

where $a^{\pm} := (\hat{p} + i\hat{q})/\sqrt{2}$ are the creation and annihilation operators, respectively.

The basis of the corresponding Hilbert space $\mathcal{H}$ is formed by the vectors of the Fock space $|n\rangle$ $(n \geq 0)$.

The spectrum for $\hat{J}_0$ reads

$$\hat{J}_0|n\rangle = (n+k)|n\rangle \tag{5.12}$$

and from (5.11) we get

$$\hat{J}_+|n\rangle = \sqrt{(n+1)(n+2k)}\,|n{+}1\rangle, \quad \hat{J}_-|n\rangle = \sqrt{n(n+2k-1)}\,|n{-}1\rangle \tag{5.13}$$

Eqs (5.10)-(5.12) present the UIR of $SO_\uparrow(2.1)$ from the discrete series of $SL(2.\mathbf{R})$ [8].

According to (5.12) the energy of a quantum particle $\hat{E} = m\hat{J}_0$ takes only discrete values $E_n = m(n+k)$, where n is a nonnegative integer.

Eq.(2.7) for $\epsilon > 0$ defines the Liouville field on a plane

$$\varphi(x^+, x^-) = -2\log m|x|$$

with singularity at $x := (x^+ - x^-)/2 = 0$. The corresponding dynamics on the half-plane $x > 0$ was considered in [1]. The global symmetry group of spacetime in this case is a group of dilatations and translations along $t := (x^+ + x^-)/2$ (similarly to the case $R_0 < 0$). However, the space of trajectories has the symmetry of $SO_\uparrow(2.1)$ group and the physical phase-space is identified with the upper-hyperboloid. Therefore, quantization in this case can be done in the same way as for the stripe.

Due to the translation invariance in time, we again have conservation of energy, but now its spectrum is continuous, since in this case $E = m(J_0 + J_1)/2$.

For the stripe (5.1) and the half-plane with $R_0 > 0$ our quantization method leads to the result (5.10), i.e., for a fixed value of space curvature $R_0 = 2m^2$, particle mass m_0 can take *only discrete* values given by

$$m_0 = m\sqrt{k(k-1)}, \qquad k = 2, 3, 4, \ldots \tag{5.14}$$

6 Conclusion

Two dimensional Lorenzian manifolds with constant curvature are locally described by the Liouville field theory, which has the symmetry associated

with $sl(2.\mathbf{R})$ algebra (see (2.6)). Therefore, the particle dynamics in the corresponding gravitational field is characterized by three dynamical integrals. Due to the mass-shell condition these integrals are on the hyperboloid (one-sheet hyperboloid for $R_0 < 0$ and upper-hyperboloid for $R_0 > 0$). There is one-to-one correspondence between the space of available dynamical integrals and the set of particle trajectories in spacetime. This set of trajectories is the physical phase-space and it depends on the global properties of spacetime. Thus, the global properties of spacetime specify the physical phase-space as an available part of the hyperboloid.

In the case when the space of particle trajectories has global $SO_{\uparrow}(2.1)$ symmetry the physical phase-space is the entire hyperboloid and our quantization method leads to the unique quantum theory with $SO_{\uparrow}(2.1)$ symmetry.

We conclude: To quantize the system it is necessary to specify (or identify) the global properties of the spacetime.

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